Lecture 6

Recall: Analytic Spectral method to solve

$$L\,u(x) = g(x). \qquad \left(\text{e.g. } L = \frac{d^2}{dx^2}\right)$$

Find basis functions $\{\phi_n(x)\}_{n=1}^{\infty}$ such that:

$$L\,\phi_n(x) = \sum_{j=1}^{N} \lambda_{j,n}\phi_j(x)$$

$$g(x) \approx \sum_{j=1}^{N} b_j\,\phi_j(x).$$

Let $u(x) = \sum_{j=1}^{N} a_j\,\phi_j(x)$.

Then: $L\,u(x) = g(x) \Rightarrow \sum_{j=1}^{N} a_j \sum_{k=1}^{N} \lambda_{k,j}\phi_k(x) = \sum_{j=1}^{N} b_j\,\phi_j(x)$

Comparing coefficients $\Rightarrow$ Diff. eqt becomes algebraic eqts.

Example: Consider $u_t - \alpha u_{xx} = h(x,t)$ such that:

$u(0,t) = u(2\pi, t) = 0$ and $u(x,0) = f(x)$.

Assuming that $h(x,t) = \sum_{k=1}^{\infty} k^2 t \sin kx$ and $f(x) = \sum_{k=1}^{\infty} k \sin kx$

Solution: This time, we consider $L = \frac{d^2}{dx^2}$.

Then: we choose $\{\phi_n(x)\}_{n=1}^{\infty} = \{\cos nx, \sin nx\}_{n=1}^{\infty}$.

We assume:

$$u(x,t) = \sum_{n=1}^{\infty} \cancel{a_n(t)\cos nx} + b_n(t)\sin nx$$

Note that $u(0,t) = u(2\pi, t) = 0$, we can remove terms with $\cos nx$

$\therefore$ we assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$.

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = h(x,t)$$

$$\Rightarrow \sum_{k=1}^{\infty} b_k'(t) \sin kx + \alpha b_k(t) k^2 \sin kx = \sum_{k=1}^{\infty} k^2 t \sin kx$$

Comparing coefficients :

$$b_k'(t) + \alpha k^2 b_k(t) = k^2 t$$

Also, $u(x,0) = \sum_{k=1}^{\infty} b_k(0) \sin kx = f(x) = \sum_{k=1}^{\infty} k \sin kx$

Comparing coefficients :

$$b_k(0) = k.$$

$\therefore$ we have:

$$\begin{cases} b_k'(t) + \alpha k^2 b_k(t) = k^2 t \\ b_k(0) = k \end{cases}$$

Can be solved using integrating factor technique:

Let $M(t) = e^{\int \alpha k^2 dt}$. Multiply both sides by $M(t)$ etc ...

Answer: $$b_k(t) = e^{-\alpha k^2 t}\left(k + \int_0^t k^2 s\, e^{\alpha k^2 s}\, ds \right)$$

(exercise!)

Recall: Many times we need to approximate $f(x)$ by:

$$f(x) = \sum_{k=0}^{N} a_k \cos kx + b_k \sin kx \quad \text{where}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \; ; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \; ; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Definition: (Real Fourier Series)

Consider $f(x) \in V = \{$ real-valued 2π-periodic smooth functions $\}$.

Then, the real Fourier Series of $f(x)$ is given by:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \,,$$ where $\{a_k\}$ and $\{b_k\}$ are given

by: $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \; ; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \; ; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Definition: (Complex Fourier Series)

Consider $f(x) \in W = \{$complex-valued 2π-periodic smooth functions$\}$

Then, the complex Fourier Series is given by:

$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{ikx}$ where $\{C_k\}$ is determined by:

$$C_k = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-\overset{\sqrt{-1}}{\overset{\shortparallel}{i}}kx}\, dx}_{\text{The integration is computed separately for the real part and imaginary part.}} \quad (\text{Here, } e^{ikx} = \cos kx + i \sin kx)$$

Example: 1. Real Fourier Series of $f(x) = \sin^2 x$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{\sin^2 x}{\overset{\shortparallel}{f(x)}}\, dx = \frac{1}{2}, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\frac{(1-\cos 2x)}{2}}{\overset{\shortparallel}{\underset{\sin^2 x}{\overset{\shortparallel}{f(x)}}}} \cos kx\, dx = \begin{cases} 0 & k \neq 2 \\ -\frac{1}{2} & k = 2 \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\frac{(1-\cos 2x)}{2}}{\overset{\shortparallel}{f(x)}} \sin kx\, dx = 0$$

$$\therefore \; f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$ (Well-known trigonometric formula)

2. Real Fourier Series of $f(x) = x$:

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right) \quad \text{for } -\pi < x < \pi$$

$$\left(a_0 = a_1 = a_2 = \cdots = 0 \;;\; b_k = (-1)^{k+1}\frac{2}{k}\right)$$

Example: Suppose $f(x) = 1$ on $[0, \pi]$.

If $f(x)$ is extended to $[-\pi, \pi]$ as an even function: $f(x) = \begin{cases} 1 & x \in [0,\pi] \\ 1 & x \in [-\pi, 0) \end{cases}$

Then, $a_0 = 1$, $a_k = b_k = 0$ for $(k \neq 0)$.

$\therefore$ Real Fourier Series of $f(x)$ is $\underline{1}$ (Recovering the original function)

If $f(x)$ is extended to $[-\pi, \pi]$ as an odd function: $f(x) = \begin{cases} 1 & x \in [0,\pi] \\ -1 & x \in [-\pi, 0) \end{cases}$

Then: $a_0 = 0$, $a_k = 0$, $(k \neq 0)$ $b_k = \begin{cases} 0 & k \text{ is even} \\ \frac{4}{k\pi} & k \text{ is odd} \end{cases}$

$\therefore$ Real Fourier Series of $f(x)$:

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \text{ for } x \in [-\pi, \pi]$$

Relationship between real and complex Fourier Series

Recall: $e^{ikx} = \cos kx + i \sin kx$; $e^{-ikx} = \cos kx - i \sin kx$.

Let $f(x) \in V$. Then: $C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx \quad (k > 0)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx$$

$$= \frac{1}{2} a_k - \frac{i}{2} b_k$$

Similarly, $C_{-k} = \frac{1}{2}(a_k + i b_k)$ and $C_0 = a_0$.

Thus, $(k > 0)$ $C_k = \overline{C_{-k}}$. (Fourier coefficients are repeated)

Also, $a_k = C_k + C_{-k}$ etc...

Question: How well does it approximate $f(x)$?

Consider: $V_N = \{ F(x) = \sum_{k=0}^{N} A_k \cos kx + B_k \sin kx : A_k, B_k \in \mathbb{R} \}$

For any 2π-periodic function, define:

$$\|f - F\|^2 := E(A_0, A_1, \dots, A_N, B_1, B_2, \dots, B_N)$$

$$:= \int_0^{2\pi} \left(f(x) - \left(\sum_{k=0}^{N} A_k \cos kx + B_k \sin kx \right) \right)^2 dx$$

Remark: $\|f - F\|$ is called the least square error between f and F.

Theorem: $E(a_0, a_1, \dots, a_N, b_1, b_2, \dots, b_N) = \min\limits_{\forall A_k, B_k \in \mathbb{R}} E(A_0, \dots, A_N, B_1, \dots, B_N)$

where:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \,; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \,; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Proof: Assume $A_0, \dots, A_N, B_1, \dots, B_N$ are the minimizer of E.

Then: $\frac{\partial E}{\partial A_i} = 0 \,; \quad \frac{\partial E}{\partial B_i} = 0.$

$$\frac{\partial E}{\partial A_k} = \frac{\partial}{\partial A_k}\int_0^{2\pi}\left(f(x) - \left(\sum_{j=0}^{N} A_j\cos jx + B_j\sin jx\right)\right)^2 dx$$

$$= -2\int_0^{2\pi}\left(f(x) - \sum_{j=0}^{N} A_j\cos jx + B_j\sin jx\right)\cos kx$$

$$= -2\int_0^{2\pi} f(x)\cos kx\,dx + 2\pi A_k = 0 \Rightarrow A_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$

Similarly, $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$ etc ...

Is this the critical point of the minimizer? HW.